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Axisymmetric Stokes flows due to a rotlet or stokeslet near a hole in a plane wall: filtration flows

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The axially symmetric Stokes-flow problems occurring when a point source, rotlet or stokeslet is situated along the axis through the centre of a circular hole in a solid plane wall are examined. Exact solutions of the governing equations are obtained in terms of toroidal coordinates and their use in modelling the flows caused by a small particle translating and rotating near to a filter pore is considered. First-order expressions are derived for the effects of the wall and hole upon the hydrodynamic force and torque on the particle for situations in which the particle dimensions are small in comparison with its distance from the solid portion of the plane wall. The resulting expressions apply to any centrally symmetric particle, not necessarily axisymmetric. Finally, expressions are derived for the motion of a neutrally buoyant sphere suspended in a flow through a hole. It is demonstrated that such a particle will generally migrate across the streamlines of the undisturbed flow – away from or towards the symmetry axis of the flow, according as the particle is approaching or receding from the hole. Such migratory motion may be of importance in the flow of suspensions through orifices and stenoses.

1. Introduction

A problem of considerable importance in filtration flows is the manner in which the suspended particles interact hydrodynamically with the pore geometry. One way of modelling the interaction between a single particle and a single pore is to regard the filter as an infinitesimally thin plane wall pierced by a small circular hole connecting semi-infinite regions of fluid on either side of it (figure 1), and to represent the suspended particle as a small rigid body, typically a sphere. We shall regard the Reynolds number based upon the particle or hole size as being small. The problem is then one of finding the Stokes resistance of the particle as it moves in the proximity of the hole in the wall.

However, even with such simplifications in the pore–particle geometry and flow, the hydrodynamic problem posed is still one of immense difficulty, seemingly intractible of mathematical solution. A further simplification which can be made is to

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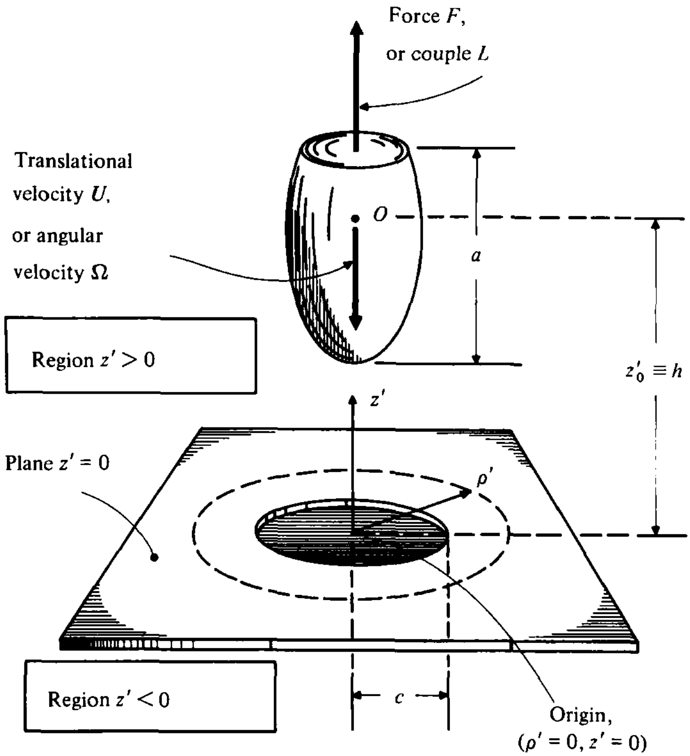

FIGURE 1. An axially symmetric particle translating or rotating relative to an axis lying normal to the plane of the wall and passing through the centre of the hole. The direction of the couple L arising from the rotation, or drag force F arising from the translation, is as indicated.

regard the particle as a point force (stokeslet) or point couple (rotlet). This approximation is sensible if the maximum linear particle dimension is small compared with the distance of the particle from the nearest solid portion of the wall.

In this paper, consideration is given to the axially symmetric flow problems occurring when a point source, rotlet or stokeslet is situated along the axis through the centre of the hole perpendicular to its plane. The problems then posed are capable of exact mathematical solution and are not without interest since each requires a different mode of solution. For each of these problems, it has been found advantageous to work with toroidal coordinates, since within this coordinate system, the hole and the plane less the hole are both members of the same family of level surfaces. The source and rotlet problems are Dirichlet boundary-value problems for the Laplace equation and modified Laplace equation respectively. Both can be solved directly using Fourier–Mehler transforms or alternatively can be solved as mixed boundary-value problems leading to the solution of dual integral equations.

The stokeslet problem is quite different. Although axial symmetry implies the existence of a Stokes stream function, it is surprisingly found that this cannot be expressed in the form of separated variables of the coordinate system. The method of solution is to set up boundary-value problems for the velocity and pressure fields which in turn can be expressed in terms of two scalar functions. An additional difficulty with this problem is that the value taken by the stream function on the plane, which is proportional to the flux of fluid through the hole, is an unknown of the problem. The method of solution employed in this paper bypasses this difficulty, and the flux of the fluid through the hole can be expressed as a simple function of the distance of the stokeslet from the plane. The reflected velocity at the point occupied by the stokeslet can also be found in a simple form, and this information utilized to compute the wall effect upon the drag force experienced by a small particle translating normal to the hole. This force has the property that it possesses a maximum value when the particle is of the order of one hole radius from the plane. Thus, a small particle moving through an otherwise quiescent fluid towards the pore of a filter experiences its greatest drag before reaching the plane of the filter pore. Furthermore, it is shown that for all positions of the stokeslet along the axis of symmetry, the force exerted by the stokeslet on the fluid is the same as the force exerted by the fluid on the plane, and this is a constant value independent of the distance of the stokeslet from the plane.

It has been established that the flux of fluid passing through the hole when a stokeslet is placed on the axis of symmetry is a non-zero quantity. This means that the flow cannot be the limit of a flow due to a stokeslet in the presence of a plane attached to a spherical bowl, where necessarily the flux through the plane is zero, giving rise to reversed flow. The solution of this problem is presented in an appendix.

2. The basic formulae

Let toroidal coordinates (ξ, η, ω) be defined (Happel & Brenner 1973) in terms of cylindrical polar co-ordinates (ρ, ω, z) by the equations

$$z = \sin\eta/(\cosh\xi - \cos\eta), \quad \rho = \sinh\xi/(\cosh\xi - \cos\eta). \tag{2.1}$$

For convenience the variables (ρ, z) have been made dimensionless with the hole radius. Later on, in §4, we shall revert to the physical (dimensional) variables (ρ', z'). The special circle is given by $\rho = 1, z = 0, |\omega| < \pi$ and is the boundary of both the unit disk $\eta = \pi, \xi \geqslant 0$ and the coincident plane surfaces $\eta = 0$ or $2\pi, \xi \geqslant 0$. All of the solutions sought here are independent of the meridional angle ω.

A solution of Laplace's equation, which is bounded everywhere, has the form

$$\phi = (\cosh\xi - \cos\eta)^{\frac{1}{2}} \int_0^\infty [A(s)\cosh s\eta + B(s)\sinh s\eta]\, K_s(\cosh\xi)\, ds, \tag{2.2}$$

where $K_s = P_{-\frac{1}{2}+is}$ is a Mehler conal function, with P_ν the Legendre function of order ν. A solution of

$$\nabla^2 w - \rho^{-2} w = 0, \tag{2.3}$$

which is bounded everywhere, is similarly of the form

$$w = (\cosh\xi - \cos\eta)^{\frac{1}{2}} \int_0^\infty [A(s)\cosh s\eta + B(s)\sinh s\eta]\, K_s^1(\cosh\xi)\, ds, \tag{2.4}$$

where $K_s^1(\cosh\xi) = \sinh\xi K_s'(\cosh\xi)$. Formulae (2.2), (2.4) are given by Schneider, O'Neill & Brenner (1973). It may appear from the differential equations (2.3) and $\nabla^2\phi = 0$ that any ϕ of the form (2.2) will be such that $\partial\phi/\partial\rho$ can be written in the form (2.4). However this is not so, owing to unboundedness as $\xi \to \infty$.

A solution of the fourth-order equation,

$$L_{-1}^2\psi \equiv \left[\frac{\partial^2}{\partial\rho^2}-\frac{1}{\rho}\frac{\partial}{\partial\rho}+\frac{\partial^2}{\partial z^2}\right]^2\psi = 0,$$

which is bounded everywhere, has, according to Payne & Pell (1960), the form

$$\psi = \rho^2(\cosh\xi-\cos\eta)^{\frac{1}{2}}\int_0^\infty F(s,\eta)\,K_s'(\cosh\xi)\,ds, \tag{2.5}$$

where F is a linear combination of the four functions $\cosh s\eta\cos\eta$, $\cosh s\eta\sin\eta$, $\sinh s\eta\cos\eta$, $\sinh s\eta\sin\eta$.

In terms of ξ and η, the square of the distance between the points (ρ, z) and $(0, z_0)$, with $z_0 = \cot\frac{1}{2}\eta_0$, is given by

$$(z-z_0)^2+\rho^2 = \frac{\cosh\xi-\cos(\eta-\eta_0)}{(\cosh\xi-\cos\eta)\sin^2\frac{1}{2}\eta_0}. \tag{2.6}$$

The formula

$$(\cosh\xi+\cos\eta_0)^{-\frac{1}{2}} = \sqrt{2}\int_0^\infty\frac{\cosh s\eta_0}{\cosh s\pi}K_s(\cosh\xi)\,ds \quad (|\eta_0|<\pi) \tag{2.7}$$

implies that

$$(\cosh\xi+\cos\eta_0)^{-\frac{3}{2}} = -2\sqrt{2}\int_0^\infty\frac{\cosh s\eta_0}{\cosh s\pi}K_s'(\cosh\xi)\,ds \quad (|\eta_0|<\pi), \tag{2.8}$$

but it is not possible to express $(\cosh\xi+\cos\eta_0)^{-\frac{1}{2}}$ in terms of an integral involving $K_s'(\cosh\xi)$. One has to manage with a difference of such functions obtained by integrating (2.8) with respect to η_0, viz.

$$\begin{aligned}(\cosh\xi+\cos\eta_1)^{-\frac{1}{2}}&-(\cosh\xi+\cos\eta_2)^{-\frac{1}{2}} \\ &= \sqrt{2}\int_0^\infty\frac{K_s'(\cosh s\xi)}{(s^2+1)\cosh s\pi}\{\cos\eta_1\cosh s\eta_1-s\sin\eta_1\sinh s\eta_1 \\ &\qquad -\cos\eta_2\cosh s\eta_2+s\sin\eta_2\sinh s\eta_2\}\,ds,\end{aligned} \tag{2.9}$$

where we also have the restriction $|\eta_1|, |\eta_2| < \pi$.

3. The source and rotlet

Suppose that ϕ satisfies Laplace's equation, has a source singularity at $(\rho = 0, z = z_0)$ where $z_0 > 0$, and takes given values at $\eta = 0, 2\pi$. Then ϕ can be written as the sum of the solution in the absence of the source and a solution $\hat{\phi}$ which vanishes at $\eta = 0, 2\pi$. Thus

$$\hat{\phi} = -[\rho^2+(z-z_0)^2]^{-\frac{1}{2}}+\sqrt{2}\sin\tfrac{1}{2}\eta_0(\cosh\xi-\cos\eta)^{\frac{1}{2}}\int_0^\infty A(s)\cosh s(\pi-\eta)\,K_s(\cosh\xi)\,ds, \tag{3.1}$$

since the additional term is obviously even in $(\pi-\eta)$. From (2.6), the vanishing of $\hat{\phi}$ at $\eta = 0, 2\pi$ requires that

$$\sqrt{2}\int_0^\infty A(s)\cosh s\pi K_s(\cosh\xi)\,ds = (\cosh\xi-\cos\eta_0)^{-\frac{1}{2}}$$

and hence, using (2.7),

$$A(s) = \cosh s(\pi-\eta_0)/\cosh^2 s\pi. \tag{3.2}$$

The Neumann problem for ϕ can be treated in a similar manner to the above Dirichlet problem.

Now suppose that w satisfies (2.3), has a rotlet singularity at $(\rho = 0, z = z_0)$ and vanishes at $\eta = 0, 2\pi$. Then

$$w = \rho\,[\rho^2 + (z-z_0)^2]^{-\frac{3}{2}} + 2\sqrt{2}\sin^3\tfrac{1}{2}\eta_0(\cosh\xi - \cos\eta)^{\frac{1}{2}}\int_0^\infty A(s)\cosh s(\pi-\eta)\,K_s^1(\cosh\xi)\,ds, \tag{3.3}$$

and the boundary condition shows, using (2.8), that $A(s)$ is again given by (3.2).

These problems can also be solved as mixed boundary value problems. Writing

$$\hat{\phi} = -[\rho^2 + (z-z_0)^2]^{-\frac{1}{2}} + \int_0^\infty e^{-k|z|}J_0(k\rho)\,B(k)\,dk, \tag{3.4}$$

the conditions which determine $B(k)$ are that $\hat{\phi} = 0$ at $z = 0, \rho > 1$ whilst $\partial\hat{\phi}/\partial z$ is continuous at $z = 0, \rho < 1$. Thus the following pair of dual integral equations is obtained:

$$\int_0^\infty J_0(k\rho)\,B(k)\,dk = (\rho^2 + z_0^2)^{-\frac{1}{2}} \quad (\rho > 1), \tag{3.5}$$

$$\int_0^\infty kJ_0(k\rho)\,B(k)\,dk = 0 \quad (\rho < 1). \tag{3.6}$$

Using the formulae (Sneddon 1966)

$$J_0(k\rho) = \frac{2}{\pi}\int_\rho^\infty \frac{\sin ks\,ds}{(s^2-\rho^2)^{\frac{1}{2}}} = \frac{2}{\pi}\int_0^\rho \frac{\cos ks\,ds}{(\rho^2-s^2)^{\frac{1}{2}}} \tag{3.7}$$

and defining $\hat{B}(s)$ to be the Fourier sine transform of $B(k)$, viz.

$$\hat{B}(s) = \int_0^\infty B(k)\sin ks\,dk, \tag{3.8}$$

(3.5) can be written as

$$\frac{2}{\pi}\int_\rho^\infty \frac{\hat{B}(s)\,ds}{(s^2-\rho^2)^{\frac{1}{2}}} = (\rho^2 + z_0^2)^{-\frac{1}{2}} \quad (\rho > 1),$$

whence, by a standard result, again given by Sneddon (1966),

$$\hat{B}(s) = -\frac{d}{ds}\int_s^\infty \frac{u\,du}{(u^2-s^2)^{\frac{1}{2}}(z_0^2+u^2)^{\frac{1}{2}}} = \frac{s}{z_0^2+s^2} \quad (s > 1).$$

Meanwhile, (3.6) can be written as

$$\int_0^\rho \frac{d\hat{B}}{ds}\frac{ds}{(\rho^2-s^2)^{\frac{1}{2}}} = 0 \quad (\rho < 1),$$

whence $\hat{B}(s) = \text{constant}$ $(s < 1)$. On substitution of these expressions for $\hat{B}(s)$ in (3.6), it emerges that the constant must be zero. Thus (3.4) becomes

$$\hat{\phi} = -[\rho^2 + (z-z_0)^2]^{-\frac{1}{2}} + \frac{2}{\pi}\int_0^\infty e^{-k|z|}J_0(k\rho)\int_1^\infty \frac{s\sin ks\,ds}{z_0^2+s^2}\,dk. \tag{3.9}$$

Similarly, writing the velocity in the rotlet problem as

$$w = \rho[\rho^2+(z-z_0)^2]^{-\frac{3}{2}} + \int_0^\infty e^{-k|z|} J_1(k\rho)\, k^{-1} C(k)\, dk \tag{3.10}$$

and defining $\hat{C}(s)$ to be the Fourier sine transform of $C(k)$, the boundary conditions imply that

$$\frac{2}{\pi}\int_0^\infty \int_0^\infty k^{-1} J_1(k\rho)\,\hat{C}(s) \sin ks\, ds\, dk = -\rho(\rho^2+z_0^2)^{-\frac{3}{2}} \quad (\rho > 1) \tag{3.11}$$

and

$$\hat{C}(s) = 0 \quad (s < 1).$$

But

$$\begin{aligned}\int_0^\infty J_1(k\rho)\, k^{-1} \sin ks\, dk &= \int_0^s dt \int_0^\infty J_1(k\rho) \cos kt\, dk \\ &= \int_0^s dt \left\{\frac{1}{\rho} - \frac{t}{\rho}\frac{H(t-\rho)}{(t^2-\rho^2)^{\frac{1}{2}}}\right\} \\ &= \frac{1}{\rho}\{s - H(s-\rho)\,(s^2-\rho^2)^{\frac{1}{2}}\},\end{aligned}$$

where H denotes the Heaviside unit function. Substituting this into (3.11) and applying the operator $\rho^{-1}\,\partial/\partial\rho$ to ρ times that equation, it follows that

$$\frac{2}{\pi}\int_\rho^\infty \frac{\hat{C}(s)\, ds}{(s^2-\rho^2)^{\frac{1}{2}}} = -2(\rho^2+z_0^2)^{-\frac{3}{2}} + 3\rho^2(\rho^2+z_0^2)^{-\frac{5}{2}} \quad (\rho > 1),$$

whence

$$\hat{C}(s) = -\frac{d}{ds}\int_s^\infty \frac{u\, du}{(u^2-s^2)^{\frac{1}{2}}}\left[-\frac{2}{(u^2+z_0^2)^{\frac{3}{2}}} + \frac{3u^2}{(z_0^2+u^2)^{\frac{5}{2}}}\right] = \frac{d}{ds}\frac{z_0^2-s^2}{(z_0^2+s^2)^2} (s > 1).$$

But since

$$\int_0^\infty J_1(k\rho) \sin ks\, dk = s\, H(\rho - s)/\rho(\rho^2-s^2)^{\frac{1}{2}},$$

it follows that an arbitrary multiple of $\sin k$ can be added to $C(k)$ without affecting the condition

$$\int_0^\infty J_1(k\rho)\, C(k)\, dk = 0 \quad (\rho > 1).$$

The sine transform of $\sin k$ is $\frac{1}{2}\pi\delta(1-s)$ and hence, with C_0 a constant to be determined,

$$\hat{C}(s) = C_0\,\delta(1-s) + H(s-1)\frac{d}{ds}\frac{z_0^2-s^2}{(z_0^2+s^2)^2}.$$

Thus

$$C(k) = \frac{2}{\pi}\left[\left(C_0 - \frac{z_0^2-1}{(z_0^2+1)^2}\right)\sin k - k\int_1^\infty \frac{z_0^2-s^2}{(z_0^2+s^2)^2}\cos ks\, ds\right].$$

The condition (3.11) now implies that

$$C_0 = -2/(z_0^2+1)^2,$$

and another integration by parts enables (3.10) to be written as

$$\begin{aligned} w &= \frac{\rho}{[\rho^2+(z-z_0)^2]^{\frac{3}{2}}} + \frac{2}{\pi}\int_0^\infty e^{-k|z|} J_1(k\rho)\left\{\frac{\cos k - k^{-1}\sin k}{(z_0^2+1)} - k\int_1^\infty \frac{s \sin ks\, ds}{z_0^2+s^2}\right\} dk \\ &= \rho[\rho^2+(z-z_0)^2]^{-\frac{3}{2}} + w_1. \end{aligned} \tag{3.12}$$

Note that, in agreement with the comments made in § 2, the expression for w in (3.12) is not the ρ-derivative of that for $\hat{\phi}$ in (3.9).

A measure of the retarding effect of the rigid boundary upon the motion can be obtained by considering the angular velocity induced by the reflected field w_1 at the rotlet. From (3.10),

$$\left(\frac{w_1}{\rho}\right)_{\substack{\rho=0\\ z=z_0}} = \frac{1}{2}\int_0^\infty J_1(k\rho)\,C(k)\,dk$$

$$= -\frac{1}{\pi}\left[\frac{1}{(z_0^2+1)^2}+\int_1^\infty \frac{(z_0^2-s^2)^2}{(z_0^2+s^2)^4}ds\right]$$

$$= -\frac{1}{\pi}\left[\frac{1}{4z_0^3}\tan^{-1}z_0-\frac{1}{4z_0^2(z_0^2+1)}+\frac{1}{2(z_0^2+1)^2}+\frac{2}{3(z_0^2+1)^3}\right]. \tag{3.13}$$

Since, from (2.1), $z_0 = \cot\frac{1}{2}\eta_0$, this expression is equivalent to that which can be obtained from (3.3), namely

$$\left(\frac{w_1}{\rho}\right)_{\substack{\rho=0\\ z=z_0}} = (1-\cos\eta_0)^3\int_0^\infty \frac{\cosh^2 s(\pi-\eta_0)}{\cosh^2 s\pi}K_s'(1)\,ds$$

$$= -\tfrac{1}{2}(1-\cos\eta_0)^3\int_0^\infty \frac{\cosh^2 s(\pi-\eta_0)}{\cosh^2 s\pi}(s^2+\tfrac{1}{4})\,ds$$

$$= -\frac{1}{8\pi}(1-\cos\eta_0)^3\left\{\frac{\cos\eta_0}{\sin^2\eta_0}+\frac{\pi-\eta_0}{\sin^3\eta_0}+\frac{2}{3}\right\} \tag{3.14}$$

The limit of this expression as the rotlet approaches the centre of the hole ($\eta_0\to\pi$) is $-4/3\pi$.

If instead of having fluid in the region $\pi<\eta<2\pi$, there is a free surface at $\eta=\pi$, then the condition $\partial w/\partial\eta = 0$ at $\eta=\pi$ can be satisfied by adding to $w(0<\eta<\pi)$ an image rotlet and its corresponding reflected field w_1, viz.

$$w = \rho[(z-z_0)^2+\rho^2]^{-\frac{3}{2}}+\rho[(z+z_0)^2+\rho^2]^{-\frac{3}{2}}+w_1(\xi,\eta,\eta_0)+w_1(\xi,\eta,2\pi-\eta_0),$$

where, evidently $w_1(\xi,\eta,2\pi-\eta_0) = w_1(\xi,\eta,\eta_0)$. If alternatively the wall has no hole but is a solid plane, then only an image rotlet of opposite sense needs to be added. It readily follows that the torque factors for a sphere of radius $a\,(\ll 1)$ are:

$$1+a^3/8z_0^3,\quad \text{solid plane;}$$

$$1-a^3\left(\frac{w_1}{\rho}\right)_{(0,z_0)},\quad \text{hole in wall;}$$

$$1-a^3/8z_0^3-2a^3\left(\frac{w_1}{\rho}\right)_{(0,z_0)},\quad \text{free surface in wall;} \tag{3.15}$$

and are in arithmetic progression. These represent the amounts by which the torque L on the sphere is increased beyond the value L_∞ that would obtain if the wall were absent and the sphere allowed to rotate with the same angular velocity.

Equation (3.13) may be employed to determine the effect of the wall (and hole) upon the couple L required to maintain the symmetric rotation of a 'small' axisymmetric particle for the configuration shown in figure 1. As a necessary preliminary we revert from the dimensionless variables employed thus far to comparable dimensional (physical) variables. Let c denote the hole radius, and define the physical variables

$$\rho' = c\rho,\quad z' = cz. \tag{3.16}$$

Denote by L_∞ the couple required to maintain the symmetric rotation, with angular velocity Ω, of a body of revolution about its symmetry axis in an *unbounded* fluid of viscosity μ. For example, for a sphere of radius a, $L_\infty = 8\pi\mu a^3\Omega$, while for a circular disk of radius a, $L_\infty = (32/3)\mu a^3\Omega$.

For a rotlet of couple strength L_∞ in the unbounded fluid the physical (tangential) velocity component w' for a singularity situated at $(\rho' = 0, z' = z_0')$ is (Brenner 1964*b*)

$$w' = wL_\infty/8\pi\mu c^3, \tag{3.17}$$

where w is the dimensionless velocity (cf. (3.12))

$$w = \rho[\rho^2+(z-z_0)^2]^{-\frac{3}{2}}, \tag{3.18}$$

with $z_0' = cz_0$. Comparison of the preceding with the singular term in (3.12) shows that the 'reflected' physical field corresponding to the above singularity is

$$w_1' = w_1 L_\infty/8\pi\mu c^3, \tag{3.19}$$

where the dimensionless field w_1 is given by the second term of (3.12).

To terms of lowest order in a/l, the wall correction factor to the couple for the present axisymmetric configuration is (Brenner 1964*b*)

$$L_\infty/L = 1-(\omega_1')_0/\Omega+O(a/l)^5, \tag{3.20}$$

wherein a is the maximum linear dimension of the particle, and l is the minimum distance of a point in the particle from the nearest solid portion of the bounding wall. In terms of characteristic distances,

$$l = \min(c, h) \tag{3.21}$$

where, as in figure 1,

$$h \equiv z_0'. \tag{3.22}$$

In (3.20),

$$(\omega_1')_0 = -\tfrac{1}{2}\lim_{\substack{\rho'\to 0\\ z'\to z_0'}}\frac{1}{\rho'}\frac{\partial}{\partial\rho'}(\rho' w_1') \tag{3.23}$$

is half the magnitude of the reflected vorticity† vector evaluated at the centre of reaction $(\rho' = 0, z' = z_0')$ of the particle. Introduction of (3.19) and (3.16) into (3.23) yields

$$(\omega_1')_0 = -\frac{1}{2}\frac{L_\infty}{8\pi\mu c^3}\left[\left(\frac{w_1}{\rho}\right)_0+\left(\frac{\partial w_1}{\partial\rho}\right)_0\right],$$

wherein the subscript 0 connotes evaluation at the singularity $(\rho = 0, z = z_0)$. Since the reflected field w_1 is necessarily analytic at all points of the fluid, and since $w_1 = 0$ everywhere along the symmetry axis $\rho = 0$, it readily follows that $(\partial w_1/\partial\rho)_0 = (w_1/\rho)_0$. Hence, (3.20) reduces to

$$\frac{L_\infty}{L} = 1+\frac{L_\infty}{8\pi\mu\Omega c^3}\left(\frac{w_1}{\rho}\right)_0+O\left(\frac{a}{l}\right)^5. \tag{3.24}$$

With use of (3.13) this may be written as

$$\frac{L_\infty}{L} = 1-\frac{L_\infty}{8\pi\mu\Omega c^3}W_0(z_0)+O\left(\frac{a}{l}\right)^5, \tag{3.25}$$

† Explicitly,

$$\omega_1' = -\hat{\mathbf{z}}\,.\,\tfrac{1}{2}(\nabla'\times\mathbf{v}_1'),\quad \text{with}\quad \mathbf{v}_1' = \hat{\boldsymbol{\varphi}}w_1',$$

in which $(\hat{\boldsymbol{\rho}}, \hat{\boldsymbol{\varphi}}, \hat{\mathbf{z}})$ are unit vectors in cylindrical polar co-ordinates.

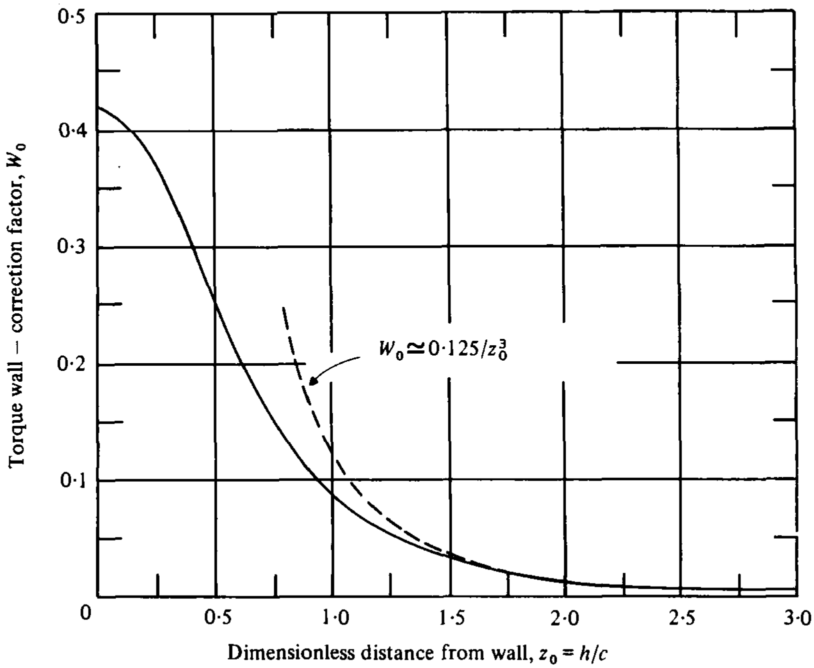

FIGURE 2. Torque correction factor W_0 for use in (3.25).

wherein

$$W_0(z_0) = \frac{1}{4\pi}\left[\frac{\tan^{-1} z_0}{z_0^3} + \frac{(3z_0^2-1)(z_0^2+3)}{3z_0^2(z_0^2+1)^3}\right] \tag{3.26}$$

is a non-dimensional function of the dimensionless ratio

$$z_0 = h/c. \tag{3.27}$$

Values of W_0 are plotted against z_0 in figure 2.

By way of example, for a spherical particle of radius a the required couple L is

$$\frac{L}{8\pi\mu a^3\Omega} = \left[1 - \left(\frac{a}{c}\right)^3 W_0(z_0) + O\left(\frac{a}{l}\right)^5\right]^{-1}. \tag{3.28}$$

The limiting case where $c \to 0$ with h fixed (i.e., $z_0 \to \infty$) corresponds to rotation near a *solid* plane wall. Since $\tan^{-1}\infty = \frac{1}{2}\pi$, we have in this limit that $W_0 \sim 1/8z_0^3 \equiv c^3/8h^3$. Hence (3.28) becomes

$$\frac{L}{8\pi\mu a^3\Omega} = \left[1 - \frac{1}{8}\left(\frac{a}{h}\right)^3 + O\left(\frac{a}{h}\right)^5\right]^{-1}. \tag{3.29}$$

This asymptotic result agrees with the leading term (Brenner 1964*b*),

$$L/8\pi\mu a^3\Omega = [1 - \tfrac{1}{8}(a/h)^3 - \tfrac{3}{256}(a/h)^8 + O(a/h)^{10}],$$

of Jeffery's (1915) exact bipolar-coordinate solution for the symmetrical rotation of a sphere near a plane wall. Equation (3.29) also applies to the case where $h/a \gg 1$ with c fixed. That is, when the sphere is sufficiently far from the wall, the hole in the wall appears to be nothing more than a pin hole, whence the couple on the sphere is the same as if the hole were totally absent.

The opposite case where $h \to 0$ with c fixed corresponds to the particle straddling the hole. Since $W_0 \sim 4/3\pi$ as $z_0 \to 0$, this makes

$$L/8\pi\mu a^3\Omega = [1-(4/3\pi)(a/c)^3+O(a/c)^5]^{-1}. \tag{3.30}$$

for a spherical particle.

4. The stokeslet

The velocity $\mathbf{v}^{(0)}$ due to the stokeslet at $(\rho = 0, z = z_0)$ has components

$$v_\rho^{(0)} = \frac{\rho(z-z_0)}{[\rho^2+(z-z_0)^2]^{\frac{3}{2}}}, \quad v_z^{(0)} = \frac{2}{[\rho^2+(z-z_0)^2]^{\frac{1}{2}}} - \frac{\rho^2}{[\rho^2+(z-z_0)^2]^{\frac{3}{2}}}, \tag{4.1}$$

while the corresponding pressure $p^{(0)}$ is given by

$$p^{(0)} = 2\mu(z-z_0)/[\rho^2+(z-z_0)^2]^{\frac{3}{2}}. \tag{4.2}$$

The problem is to calculate the additional velocity field due to the presence of the wall with a hole and it is possible to do so in terms of harmonic functions. The appropriate representation of the total velocity field $\mathbf{v}$ is

$$\mathbf{v} = \mathbf{v}^{(0)}+\mathbf{v}^{(1)}+\mathbf{v}^{(2)}, \tag{4.3}$$

where

$$\mathbf{v}^{(1)} = z\,\mathrm{grad}\,\phi - \phi\hat{\mathbf{z}}, \tag{4.4a}$$

$$\mathbf{v}^{(2)} = z\,\mathrm{grad}\,(\partial\chi/\partial z) - (\partial\chi/\partial z)\,\hat{\mathbf{z}} + \mathrm{grad}\,\chi, \tag{4.4b}$$

with

$$\nabla^2\phi = 0 = \nabla^2\chi.$$

The total pressure field p is then given by

$$p = p_\infty + p^{(0)} + 2\mu\left\{\frac{\partial\phi}{\partial z}+\frac{\partial^2\chi}{\partial z^2}\right\}.$$

The condition that $\mathbf{v}$ vanish on the wall means that

$$\phi = v_z^{(0)} = 2(\rho^2+z_0^2)^{-\frac{1}{2}} - \rho^2(\rho^2+z_0^2)^{-\frac{3}{2}} \quad (z = 0, \rho > 1) \tag{4.5}$$

and

$$\partial\chi/\partial\rho = -v_\rho^{(0)}, \quad \text{i.e.} \quad \chi = -z_0(\rho^2+z_0^2)^{-\frac{1}{2}} \quad (z = 0, \rho > 1). \tag{4.6}$$

Evidently ϕ and χ are both symmetric about the plane of the wall and have, as described in § 2, solutions in terms of toroidal co-ordinates of the forms

$$\phi = (\cosh\xi - \cos\eta)^{\frac{1}{2}} \int_0^\infty D(s)\cosh s(\pi-\eta)\,K_s(\cosh\xi)\,ds, \tag{4.7}$$

$$\chi = (\cosh\xi - \cos\eta)^{\frac{1}{2}} \int_0^\infty E(s)\cosh s(\pi-\eta)\,K_s(\cosh\xi)\,ds. \tag{4.8}$$

Condition (4.6) means that the solution for $E(s)$ is a constant multiple of $A(s)$ given by (3.2), the result being

$$E(s) = -\sqrt{2}\cos\tfrac{1}{2}\eta_0 \frac{\cosh s(\pi-\eta_0)}{\cosh^2 s\pi}. \tag{4.9}$$

Meanwhile (4.5) implies that

$$(\cosh\xi-1)^{\frac{1}{2}}\int_0^\infty D(s)\cosh s\pi K_s(\cosh\xi)\,ds = \frac{2}{(\rho^2+z_0^2)^{\frac{1}{2}}}-\frac{\rho^2}{(\rho^2+z_0^2)^{\frac{3}{2}}}.$$

Substituting (2.6) into the right-hand side and using (2.7) and its η_0 derivative, it follows that

$$\begin{aligned} D(s) &= \surd 2\sin\tfrac{1}{2}\eta_0[(2-\sin^2\tfrac{1}{2}\eta_0)\cosh s(\pi-\eta_0)-s\sin\eta_0\sinh s(\pi-\eta_0)]/\cosh^2 s\pi \\ &= \surd 2(1+\cos\eta_0)\frac{\partial}{\partial\eta_0}\left\{\frac{\sin^2\frac{1}{2}\eta_0\cosh s(\pi-\eta_0)}{\cos\frac{1}{2}\eta_0\cosh^2 s\pi}\right\}. \end{aligned} \tag{4.10}$$

The flux of fluid through the hole is given, from (4.3), by

$$\begin{aligned} M &= 2\pi\int_0^1 \{v_z^{(0)}-\phi\}_{z=0}\,\rho\,d\rho \\ &= 2\pi\int_0^1\left\{\frac{2\rho}{(\rho^2+z_0^2)^{\frac{1}{2}}}-\frac{\rho^3}{(\rho^2+z_0^2)^{\frac{3}{2}}}\right\}\rho\,d\rho-\pi\int_0^\infty \tanh\tfrac{1}{2}\xi\,\mathrm{sech}^2\tfrac{1}{2}\xi(\phi)_{\eta=\pi}\,d\xi \\ &= \frac{2\pi}{(1+z_0^2)^{\frac{1}{2}}}-2\pi\int_0^\infty D(s)\int_1^\infty\frac{K_s(x)\,dx}{(x+1)^{\frac{3}{2}}}\,ds \\ &= 2\pi\sin\tfrac{1}{2}\eta_0-8\pi(1+\cos\eta_0)\frac{\partial}{\partial\eta_0}\left\{\frac{\sin^2\frac{1}{2}\eta_0}{\cos\frac{1}{2}\eta_0}\int_0^\infty\left[\frac{2\cosh s\eta_0}{\sinh 2s\pi}-\frac{\sinh s\eta_0}{\cosh^2 s\pi}\right]s\,ds\right\} \\ &= 2\pi\sin\tfrac{1}{2}\eta_0-2(1+\cos\eta_0)\frac{\partial}{\partial\eta_0}\{\pi(\sec\tfrac{1}{2}\eta_0-1)-2\tan\tfrac{1}{2}\eta_0+\eta_0\} \\ &= 4\sin^2\tfrac{1}{2}\eta_0 \\ &= \frac{4}{(1+z_0^2)}, \end{aligned}$$

after substituting (4.7) and using the formulae

$$\int_1^\infty\frac{K_s(x)\,dx}{(x+1)^{\frac{3}{2}}}=\frac{2\surd 2\,s}{\sinh s\pi},\quad \int_0^\infty\frac{s\cosh s\eta_0}{\sinh 2s\pi}ds=\frac{1}{16}\sec^2\left(\frac{\eta_0}{4}\right),$$

$$\int_0^\infty\frac{s\sinh s\eta_0}{\cosh^2 s\pi}ds=\frac{1}{2\pi}\mathrm{cosec}\,\tfrac{1}{2}\eta_0-\frac{\eta_0}{4\pi}\mathrm{cosec}\,\tfrac{1}{2}\eta_0\cot\tfrac{1}{2}\eta_0.$$

An alternative approach is to use a stream function ψ defined by

$$v_\rho=\rho^{-1}\partial\psi/\partial z,\quad v_z=-\rho^{-1}\partial\psi/\partial\rho. \tag{4.11}$$

Then $L_{-1}^2\psi=0$ and if ψ vanishes on the axis $z=0$, it must be non zero on the wall, namely

$$(\psi)_{\substack{z=0\\ \rho>1}}=-M/2\pi=-2/\pi(1+z_0^2). \tag{4.12}$$

This non-vanishing property makes it impossible to construct a solution for ψ of the form (2.5). A way of circumventing this difficulty might be to use (2.5) to find a stream function vanishing on the wall, calculate the corresponding pressure difference ΔP between $z\to-\infty$ and $z\to+\infty$ and add a suitable multiple of the solution given by Happel & Brenner (1973, §4.29) in terms of oblate spheroidal coordinates, for the pressure driven flow through a circular orifice. The idea, of course, is to cancel ΔP and hence determine the unknown 'flux' constant M. However, the method is unsuccessful because, after much algebra, ΔP is found to be zero. Further remarks on the physical significance of this fact are offered in §5.

To examine the reason for this failure, consider the components of stream function corresponding to the velocity representation (4.3). Comparing (4.11) with (4.1) and (4.4) it readily follows that

$$\psi^{(0)} = -\rho^2[\rho^2+(z-z_0)^2]^{-\frac{1}{2}}, \quad \psi^{(2)} = z\rho\, \partial\chi/\partial\rho,$$

but considerable manipulation of (4.7) is required for an expression for $\psi^{(1)}$, which turns out to be

$$\psi^{(1)} = \int_0^\infty D(s)\cosh s(\pi-\eta)\int_1^{\cosh\xi}\left[\frac{\frac{3}{2}\sin^2\eta}{(x-\cos\eta)^{\frac{5}{2}}}-\frac{\cos\eta}{(x-\cos\eta)^{\frac{3}{2}}}\right]K_s(x)\,dx\,ds$$
$$-\sin\eta\int_0^\infty sD(s)\sinh s(\pi-\eta)\int_1^{\cosh\xi}\frac{K_s(x)}{(x-\cos\eta)^{\frac{3}{2}}}\,dx\,ds,$$

where $D(s)$ is given by (4.10). These results verify that it was wrong to seek a solution of the form (2.5) for ψ, when there is a net flux of fluid through the orifice. This difficulty does not arise if the fluid on the opposite side from the stokeslet is of bounded extent, and the solution for the case when the boundary is $\eta = \pi+\eta_1 (0 < \eta_1 < \pi)$ is given in the appendix. It is of interest that the limit of this solution as $\eta_1 \to \pi$ cannot recover the solution given in this section.

Having found integral expressions for the functions ϕ, χ in the representation (4.4), the physical problem demands particular consideration of the 'reflected velocity' at the stokeslet which, from (4.3), (4.4), is given by

$$-\left(\phi - z\frac{\partial\phi}{\partial z} - z\frac{\partial^2\chi}{\partial z^2}\right)_{\substack{\rho=0\\ z=z_0}}\hat{\mathbf{z}} = -V_0\hat{\mathbf{z}}, \quad \text{say.} \tag{4.13}$$

Setting $\rho = 0\,(=\xi)$ in the subsequent calculation, it follows from (4.7), (4.10) that, since $K_s(1) = 1$,

$$\phi = (1-\cos\eta)^{\frac{1}{2}}\int_0^\infty D(s)\cosh s(\pi-\eta)\,ds,$$

$$\left(\phi - z\frac{\partial\phi}{\partial z}\right)_{z=z_0} = \left(\phi+\sin\eta\frac{\partial\phi}{\partial\eta}\right)_{\eta=\eta_0}$$
$$= \tfrac{1}{4}(1-\cos\eta_0)\int_0^\infty ds$$
$$\times \{(3+\cos\eta_0)\cosh s(\pi-\eta_0) - 2s\sin\eta_0\sinh s(\pi-\eta_0)\}^2/\operatorname{cosech}^2 s\pi$$
$$= \frac{1}{8\pi}(1-\cos\eta_0)\{3+6\cos\eta_0+\cos^2\eta_0-\tfrac{1}{3}\sin^2\eta_0+10[(\pi-\eta_0)/\sin\eta_0]\}$$

by evaluating the integrals. Also, from (4.8), (4.9):

$$-\chi = 2\sin\tfrac{1}{2}\eta\cos\tfrac{1}{2}\eta_0\int_0^\infty\frac{\cosh s(\pi-\eta_0)\cosh s(\pi-\eta)}{\cosh^2 s\pi}\,ds$$
$$= \sin\tfrac{1}{2}\eta\cos\tfrac{1}{2}\eta_0\left[\left(1-\frac{\eta+\eta_0}{2\pi}\right)\operatorname{cosec}\left(\frac{\eta+\eta_0}{2}\right)+\frac{\eta-\eta_0}{2\pi}\operatorname{cosec}\left(\frac{\eta-\eta_0}{2}\right)\right].$$

It then follows that

$$-\left(z\frac{\partial^2\chi}{\partial z^2}\right)_{z=z_0} = -\sin^2\eta_0\left(\frac{\partial\chi}{\partial\eta}\right)_{\eta=\eta_0} - \sin\eta_0(1-\cos\eta_0)\left(\frac{\partial^2\chi}{\partial\eta^2}\right)_{\eta=\eta_0}$$
$$= \frac{\sin^2\eta_0}{4\pi}\left(\frac{\pi-\eta_0}{\sin\eta_0}+\cos\eta_0\right) - \frac{\sin\eta_0}{4\pi}(1-\cos\eta_0)$$
$$\times\left\{\left(\frac{\pi-\eta_0}{\sin\eta_0}\right)\cot\eta_0 + \tfrac{1}{3}\sin\eta_0 + \operatorname{cosec}\eta_0\right\}$$
$$= \frac{(1-\cos\eta_0)}{4\pi}\left[\frac{\pi-\eta_0}{\sin\eta_0}+\cos\eta_0 - \tfrac{4}{3}\sin^2\eta_0\right].$$

On substituting these formulae into (4.13), it is seen that the reflected velocity at the stokeslet has magnitude V_0 given by

$$V_0(\eta_0) = \frac{(1-\cos\eta_0)}{2\pi}\left[3\left(\frac{\pi-\eta_0}{\sin\eta_0}\right) + 2\cos\eta_0 + \cos^2\eta_0\right\} \tag{4.14}$$

and direction opposite to that of the stokeslet.

Also of physical interest is the balance of forces acting on the fluid. Consider first the stokeslet in isolation and let spherical polar coordinates by defined by

$$z - z_0 = r\cos\theta, \quad \rho = r\sin\theta.$$

Then, from (4.1), the radial and transverse velocity components are

$$v_r^{(0)} = 2\cos\theta/r, \quad v_\theta^{(0)} = -\sin\theta/r.$$

The normal stress $T_{rr}^{(0)}$ is then given by

$$T_{rr}^{(0)} = -p^{(0)} + 2\mu\frac{\partial v_r^{(0)}}{\partial r} = -\frac{6\mu}{r^2}\cos\theta,$$

while $T_{r\theta}^{(0)}$ vanishes identically. Hence the force exerted by the stokeslet on the fluid is $8\pi\mu$ in the positive z-direction. Evidently this is the force exerted on the remaining fluid by any mass of fluid containing the stokeslet.

Now consider again the stokeslet in the presence of the hole in the plane. Since ϕ, χ are even functions of $(\eta-\pi)$ and hence of z, the z-component of the force exerted by the fluid on the plane is evidently

$$-4\pi\mu\int_1^\infty\left(\frac{\partial\phi}{\partial z}\right)_{z=0-}^{z=0+}\rho\,d\rho = -8\pi\mu\int_0^\infty\left(\frac{\partial\phi}{\partial z}\right)_{z=0+}\frac{(\cosh\xi+1)^{\frac{1}{2}}\,d\xi}{(\cosh\xi-1)^{\frac{3}{2}}}$$
$$= 8\pi\mu\int_0^\infty sD(s)\sinh s\pi\int_1^\infty\frac{K_s(x)\,dx}{(x-1)^{\frac{1}{2}}}ds$$

on substitution of (4.7). The x-integral is given by Schneider, O'Neill & Brenner (1973) and thus the double integral reduces to

$$8\sqrt{2}\,\pi\mu\int_0^\infty D(s)\cosh s\pi\,ds.$$

When (4.10) is substituted into this integrand, the algebra becomes elementary and yields the value $8\pi\mu$. Thus it is found that for all positions of the stokeslet on the axis, i.e. all z_0, the force exerted by the stokeslet on the fluid is equal to the force exerted by the fluid on the plane.

Since the force integral over the sphere at infinity is now zero, it appears that the leading terms, at large distance, of $\mathbf{v}^{(1)}$ cancel $\mathbf{v}^{(0)}$ in the representation (4.3). A simple examination of (4.7) for ϕ readily shows this to be the case. At large distances $\xi^2+\eta^2 \to 0$, whence

$$\phi \sim \left(\frac{\xi^2+\eta^2}{2}\right)^{\frac{1}{2}} \int_0^\infty D(s)\cosh s\pi\, ds = \left(\frac{\xi^2+\eta^2}{4}\right)^{\frac{1}{2}} \sim [\rho^2+(z-z_0)^2]^{-\frac{1}{2}}.$$

Hence, on substituting in (4.3) and comparing with (4.1), it is seen that

$$\mathbf{v}^{(1)} \sim -\mathbf{v}^{(0)} \quad \text{as} \quad r \to \infty.$$

Equation (4.14) may be employed to determine the effect of the wall (and hole) upon the force F required to maintain the symmetric translation of a 'small' axisymmetric particle for the configuration shown in figure 1. The technique (Sonshine, Cox & Brenner 1966) for obtaining the first-order wall effect F/F_∞ for translation from the fundamental solution for a stokeslet is wholly analogous to that employed in §3 for determining the first-order wall effect L/L_∞ for rotation from the fundamental solution for a rotlet.

The physical velocity and pressure fields $(\mathbf{v}', p')$ for a point force of strength F_∞ situated at the point $(\rho' = 0, z' = z_0')$ in an unbounded fluid of viscosity μ may be obtained from (4.1) and (4.2) by replacing (ρ, z) by (ρ', z') as in (3.16), and multiplying by $F_\infty/8\pi\mu$ (Happel & Brenner 1973). Analogous to (3.20) the wall correction factor F/F_∞ for the force experienced by a particle translating with velocity U is (Sonshine *et al.* 1966)

$$\frac{F_\infty}{F} = 1+(v_{z1}')_0/U+O(a/l)^3, \tag{4.15}$$

where $\mathbf{v}_1'$ is the reflected velocity field. From (4.13) we obtain

$$(v_{z1}')_0 = -V_0F_\infty/8\pi\mu c,$$

with $V_0(\eta_0) \equiv V_0(z_0)$ the function defined in (4.14), in which $z_0 = \cot(\eta_0/2) \equiv h/c$. This makes

$$\frac{F_\infty}{F} = 1-V_0F_\infty/8\pi\mu cU+O(a/l)^3, \tag{4.16}$$

analogous to (3.25). Equation (4.14) may be expressed alternatively in terms of z_0 by noting that

$$z_0 = \cot(\eta_0/2) = \tan[(\pi-\eta_0)/2], \quad \text{whence} \quad \pi-\eta_0 = 2\tan^{-1}z_0.$$

Consequently,

$$V_0(z_0) = \frac{1}{\pi}\left[\frac{3\tan^{-1}z_0}{z_0}+\frac{(3z_0^2+1)(z_0^2-1)}{(z_0^2+1)^3}\right]. \tag{4.17}$$

By way of example, $F_\infty = 6\pi\mu aU$ for a sphere of radius a and $F_\infty = 16\mu aU$ for a

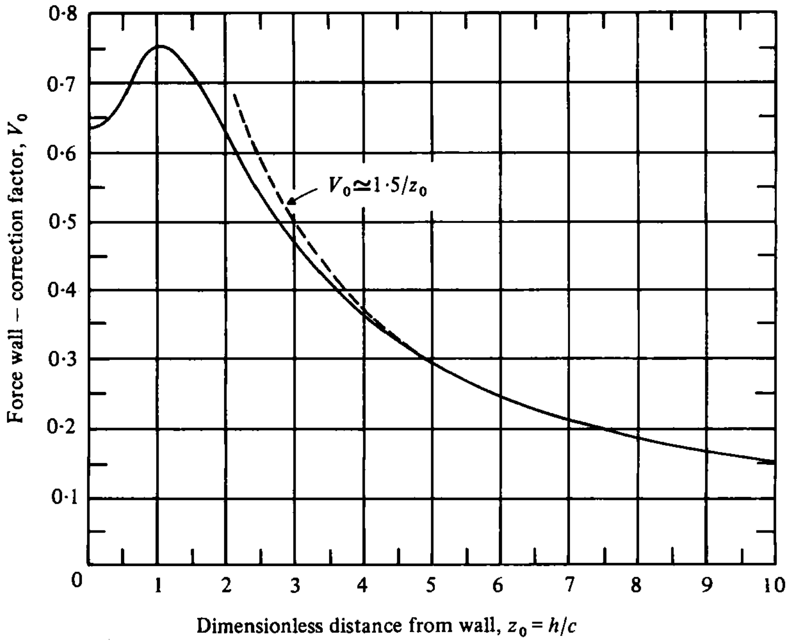

FIGURE 3. Force correction factor V_0 for use in (4.16).

circular disk of radius a moving broadside on.† Thus, for the special case of a sphere, (4.16) becomes

$$F/6\pi\mu a U = [1 - \tfrac{3}{4}(a/c)\, V_0(z_0) + O(a/l)^3]^{-1}. \tag{4.18}$$

In the limit where $c \to 0$ (with h fixed) or $h \to \infty$ with c fixed, $V_0 \sim 3/2z_0 \equiv 3c/2h$. Therefore, (4.18) becomes

$$F/6\pi\mu a U = [1 - \tfrac{9}{8}(a/h) + O(a/h)^3]^{-1}. \tag{4.19}$$

This asymptotic result agrees with the leading term of Brenner's (1961) exact bipolar coordinate solution for the translation of a sphere normal to a plane wall.

Values of the wall-effect parameter V_0 are plotted as a function of $z_0 = h/c$ in figure 3. Special values of interest are: at $z_0 = 0$, $V_0 = 2/\pi$; at $z_0 = 1$, $V_0 = 3/4$; at $z_0 = \infty$, $V_0 = 0$. The function V_0 has two turning values, the obvious one at $z_0 = 0$ and a maximum when z_0 is slightly greater than unity. This behaviour accords with intuition. When the body is situated at large distances from the hole, the hole appears to be little more than a pinhole. Accordingly, the wall appears to be solid, whence the resistance of the particle increases monotonically as it moves toward the plane. On the other hand when the particle is near the plane its resistance necessarily decreases monotonically as it gets closer to the centre of the hole, since the effect of the *solid* portion of the plane is necessarily diminished as the particle 'sees' more and more of the free expanse of the

† Equation (4.16) is not limited to axisymmetric particles (Brenner 1962*a*, 1964*a*; Cox & Brenner 1967), but may be applied to any centrally symmetric particle translating parallel to a principal axis of translational resistance along the normal to the plane of the wall containing the centre of the hole. Thus, for example, (4.16) may be applied to the case of a circular disk (of radius a) translating edge-on, for which $F_\infty = (32/3)\,\mu a U$. This edge-on configuration does not correspond to an axisymmetric motion.

fluid on the other side of the hole. Thus, the existence of a maximum somewhere in the neighbourhood of $z_0 \approx 1$ appears entirely appropriate on physical grounds.

5. Discussion

As shown in §4, the translational motion of the particle gives rise to a net flux through the hole. In physical variables the volumetric flow rate Q through the hole in the direction of z' negative is related to the dimensionless flux parameter M via the expression

$$Q = (F_\infty c/8\pi\mu)\, M.$$

Consequently,

$$Q = \frac{F_\infty c^3}{2\pi\mu}\frac{1}{c^2+h^2}. \tag{5.1}$$

Alternatively, if $\bar{V} = Q/\pi c^2$ is the mean velocity of flow through the hole,

$$\bar{V} = \frac{F_\infty c}{2\pi^2\mu}\frac{1}{c^2+h^2}. \tag{5.2}$$

This velocity properly vanishes in the limit where either $F_\infty = 0$, $c = 0$, or $h = \infty$. In the particular case of a spherical particle of radius a this mean velocity is

$$\bar{V} = U3ac/(c^2+h^2). \tag{5.3}$$

It is important to note that this represents the flow when the pressure at infinity in each chamber on either side of the hole has the same value, i.e. when $\Delta P = 0$, where

$$\Delta P = p'(z' \to -\infty) - p'(z \to +\infty), \tag{5.4}$$

(cf. the remarks following (4.12)). This distinguishes the flux from that which obtains when a non-zero externally applied pressure difference ΔP is maintained across the hole, namely (Happel & Brenner 1973)

$$q = \Delta P c^3/3\mu, \tag{5.5}$$

the direction of net flow being in the direction of diminishing pressure.

Since real experiments are invariably performed in bounded systems, there can be no net flow through the hole in such situations. Thus, putting $|q| = |Q|$ we find that the pressure is larger in that chamber from which the particle is absent than it is in the chamber containing the particle, this pressure difference being

$$\Delta P = 3F_\infty/2\pi(c^2+h^2). \tag{5.6}$$

The additional flow q across the hole from $z' < 0$ to $z' > 0$ caused by this pressure difference will exert an additional drag force on the particle above and beyond the force F given by (4.16). The additional force F^+ may be calculated to terms of lowest order in a/l by means of Faxen's law (Happel & Brenner 1973), which in present circumstances adopts the form

$$F^+ = F_\infty(v_z^+)_0/U + O(a/l)^3. \tag{5.7}$$

where v_z^+ is the physical velocity component in the z direction for the externally-driven flow through a circular hole in the wall, and 0 denotes evaluation at the point of the homogeneous fluid occupied by the centre of the particle. The error term of $O(a/l)^3$ in (5.7) arises from our neglect of the pressure gradient term in Faxen's law.

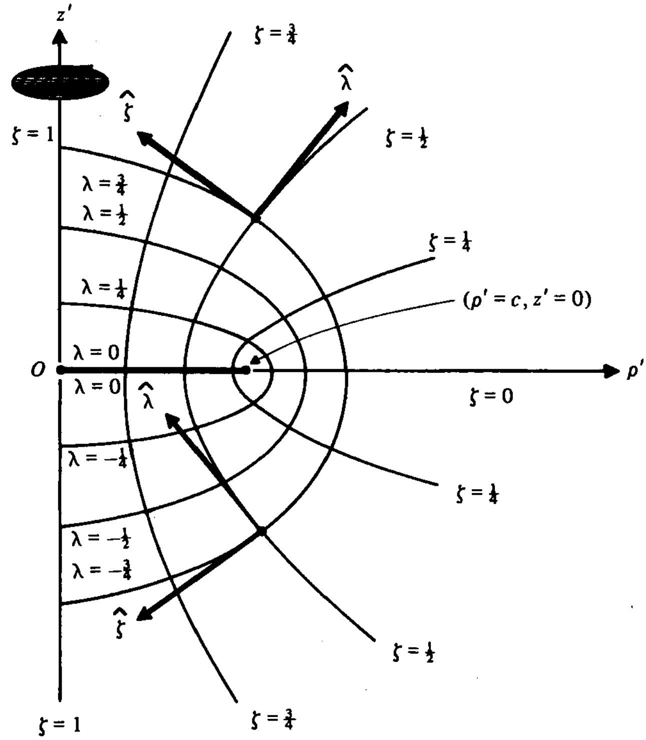

FIGURE 4. Oblate spheroidal co-ordinates (λ, ζ) in a meridian plane, $\omega =$ const. The surfaces of revolution $\lambda =$ const. are a confocal family of oblate hemispheroids having the z' axis as their common axis of revolution. As indicated in the sketch the pair of hemispheroids $\pm\lambda$ define the surface of a complete spheroid. The focal circle of the confocal family lies in the plane $z' = 0$ and corresponds to the value $\rho' = c$. The spheroid $\lambda = 0$ is degenerate, and corresponds to that portion, $0 \leqslant \rho' \leqslant c$, of the plane $z' = 0$ lying within the focal circle. The co-ordinate surfaces $\zeta =$ const. are a family of confocal hyperboloids of revolution of one sheet, having the z' axis as their common axis of revolution. For $\zeta = 1$ the hyperboloid degenerates into the line $\rho' = 0$, corresponding to the entire z' axis $(-\infty < z' < \infty)$. The value $\zeta = 0$ is also a degenerate hyperboloid, corresponding to that portion, $c \leqslant \rho' < \infty$, of the plane $z' = 0$ external to the focal circle. Typical unit vectors $(\hat{\boldsymbol{\lambda}}, \hat{\boldsymbol{\zeta}})$ are as shown in the sketch. The unit vector $\hat{\boldsymbol{\omega}}$ is directed into the plane of the page.

Happel & Brenner (1973, §4.29) give the stream function ψ^+ and pressure field p^+ for flow in the positive z' direction through a circular hole in the wall at volumetric flow rate q as

$$\psi^+ = -\frac{q}{2\pi}(1-\zeta^3) \tag{5.8}$$

and

$$p^+ = \text{const} - \frac{3\mu q}{\pi c^3}\left(\frac{\lambda}{\lambda^2+\zeta^2} + \tan^{-1}\lambda\right). \tag{5.9}$$

In these expressions (ζ, λ, ω) are a system of oblate spheroidal co-ordinates, right-handed in the order specified (figure 4). They are related to the cylindrical polar system (ρ', ω, z') via the expressions

$$\rho' = c(\lambda^2+1)^{\frac{1}{2}}(1-\zeta^2)^{\frac{1}{2}}, \quad z' = c\lambda\zeta, \tag{5.10}$$

and span the range $0 \leqslant \zeta \leqslant 1$, $-\infty < \lambda < \infty$. The circular hole corresponds to the value $\lambda = 0$, the remainder of the plane $z' =$ constant to $\zeta = 0$, and the positive and negative z' axes to $\zeta = 1$. Metrical coefficients for the system (Happel & Brenner 1973, p. 490) are

$$h_\zeta = \frac{1}{c}\left(\frac{1-\zeta^2}{\lambda^2+\zeta^2}\right)^{\frac{1}{2}}, \quad h_\lambda = \frac{1}{c}\left(\frac{\lambda^2+1}{\lambda^2+\zeta^2}\right)^{\frac{1}{2}}, \quad h_\omega = \frac{1}{\rho'}. \tag{5.11}$$

The general relationship between the velocity components in this system and the stream function is (Happel & Brenner 1973, §4.3)

$$v_\zeta^+ = -\frac{1}{c^2[(1-\zeta^2)(\lambda^2+\zeta^2)]^{\frac{1}{2}}}\frac{\partial\psi^+}{\partial\lambda}, \quad v_\lambda^+ = \frac{1}{c^2[(\lambda^2+1)(\lambda^2+\zeta^2)]^{\frac{1}{2}}}\frac{\partial\psi^+}{\partial\zeta}. \tag{5.12}$$

The velocity component v_z^+ in the z direction may be obtained from the relationship between the unit vectors $\hat{\mathbf{z}}$ and $(\hat{\boldsymbol{\zeta}}, \hat{\boldsymbol{\lambda}})$, namely

$$\hat{\mathbf{z}} = \hat{\boldsymbol{\zeta}}\lambda\left(\frac{1-\zeta^2}{\lambda^2+\zeta^2}\right)^{\frac{1}{2}} + \hat{\boldsymbol{\lambda}}\zeta\left(\frac{\lambda^2+1}{\lambda^2+\zeta^2}\right)^{\frac{1}{2}}. \tag{5.13}$$

This is readily derived by observing that $\hat{\mathbf{z}} = \nabla' z'$, writing ∇' in the oblate spheroidal co-ordinate system (ζ, λ), and using the second of equations (5.10). From (5.13) we may establish the general relation

$$v_z^+ = v_\zeta^+\,\lambda\left(\frac{1-\zeta^2}{\lambda^2+\zeta^2}\right)^{\frac{1}{2}} + v_\lambda^+\,\zeta\left(\frac{\lambda^2+1}{\lambda^2+\zeta^2}\right)^{\frac{1}{2}} \tag{5.14}$$

connecting the velocity components in the two different co-ordinate systems.

From (5.8) and (5.12) there follows

$$v_\zeta^+ = 0, \quad v_\lambda^+ = \frac{3q\zeta^2}{2\pi c^2[(\lambda^2+1)(\lambda^2+\zeta^2)]^{\frac{1}{2}}}. \tag{5.15}$$

The singular point $(\rho' = 0, z' = z_0')$ corresponds to the values $(\zeta = 1, \lambda = z_0)$ with $z_0 = z_0'/c = h/c$. Hence, from (5.14) and (5.15), at the singular point

$$(v_z^+)_0 = \frac{3q}{2\pi c^2(1+z_0^2)}. \tag{5.16}$$

This expression may be substituted into (5.7), and the parameter $q \equiv Q$ eliminated via (5.1) to obtain the additional drag-force wall correction.

$$\frac{F^+}{F_\infty} = \frac{F_\infty}{\mu U}\frac{3}{4\pi^2 c(1+z_0^2)}, \tag{5.17}$$

acting in the positive z' direction, opposite to the direction of motion U of the particle.

The total force F^* acting on the particle is $F^* = F + F^+$. Hence, with use of (4.16) we obtain the wall-effect correction factor

$$\frac{F^*}{F_\infty} = 1 + \frac{F_\infty}{8\pi\mu c U}\left[V_0(z_0) + \frac{6}{\pi(1+z_0^2)}\right] + O\left(\frac{a}{l}\right)^2 \tag{5.18}$$

for the case of no net flow through the hole. This differs from the zero pressure-difference case (4.16), i.e.

$$\frac{F}{F_\infty} = 1 + V_0(z_0)\,F_\infty/8\pi\mu c U + O\,(a/l)^2, \tag{5.19}$$

by the intrusion of the additional term in (5.18).

That the force is distinctly different for the zero flow and zero pressure-difference cases is analogous to the comparable conclusion reached for the case of a thin circular wire translating parallel to itself at the centre of a concentric circular tube filled with viscous fluid (Brenner 1962*b*). In that case too the force (per unit length of wire) was greater for the zero flow case than for the zero pressure-gradient case. In the latter case the fluid is dragged along by the wire to which it adheres, resulting in a net flow of fluid.

None of the qualitative conclusions of §4 pertaining to the wall correction factor F/F_∞ are altered upon replacing the latter by F^*/F_∞.

6. Transport of a neutrally buoyant sphere by a flow through a circular hole in a wall

Of special interest in filtration applications is the case of a *neutrally buoyant* particle, subject to no net forces or torques, being carried along by a pressure-driven flow through a hole in a circular wall. To terms of dominant order in a/l the translational velocity vector $\mathbf{U}_0$ of the centre O of a neutrally buoyant spherical particle suspended in the undisturbed flow field $(\mathbf{v}^+, p^+)$ of §5 is readily determined from Faxen's law (Happel & Brenner 1973) to be

$$\mathbf{U}_0 = \mathbf{v}_0^+ + (a^2/6\mu)\,(\nabla' p^+)_0 + O(a/l)^3. \tag{6.1}$$

(To this same order in a/l the sphere rotates with the same angular velocity as does a fluid particle situated at 0.)

Define the slip-velocity vector $\mathbf{U}^s = \mathbf{U}_0 - \mathbf{v}_0^+$ of the sphere centre. This vector,

$$\mathbf{U}^s = (a^2/6\mu)\,(\nabla' p^+)_0 + O(a/l)^3 \tag{6.2}$$

therefore represents the velocity of the sphere centre relative to the surrounding fluid. From (5.15) the local fluid velocity vector for flow through the hole in the positive z direction at mean velocity $\overline{V} = q/\pi c^2$ is

$$\mathbf{v}^+ = \hat{\boldsymbol{\lambda}} v_\lambda^+, \tag{6.3}$$

wherein

$$v_\lambda^+ = \frac{3\overline{V}}{2}\zeta^2[(\lambda^2+1)(\lambda^2+\zeta^2)]^{-\frac{1}{2}}. \tag{6.4}$$

Since the scalar v_λ^+ is non-negative, the direction of the vector $\mathbf{v}$ accords with the sketch in figure 4 depicting the direction of the unit vector $\hat{\boldsymbol{\lambda}}$.

With p^+ given by (5.9) the pressure gradient required in (6.2),

$$\nabla' p^+ = \hat{\boldsymbol{\zeta}} h_\zeta\, \partial p/\partial\zeta + \hat{\boldsymbol{\lambda}} h_\lambda\, \partial p/\partial\lambda,$$

is easily calculated. The slip-velocity vector may be resolved into the vector components $\mathbf{U}_\lambda^s = \hat{\boldsymbol{\lambda}} U_\lambda^s$ and $\mathbf{U}_\zeta^s = \hat{\boldsymbol{\zeta}} U_\zeta^s$ along and across the streamlines, respectively. In these expressions, we have the algebraically-signed scalars,

$$U_\lambda^s = \left(\frac{a}{c}\right)^2 \frac{\overline{V}}{2}\left(\frac{\lambda^2+1}{\lambda^2+\zeta^2}\right)^{\frac{1}{2}}\left[\frac{\lambda^2-\zeta^2}{(\lambda^2+\zeta^2)^2} - \frac{1}{\lambda^2+1}\right] + O\left(\frac{a}{l}\right)^3, \tag{6.5}$$

and

$$U_\zeta^s = \left(\frac{a}{c}\right)^2 \frac{\overline{V}}{2}\left(\frac{1-\zeta^2}{\lambda^2+\zeta^2}\right)^{\frac{1}{2}} \frac{2\lambda\zeta}{(\lambda^2+\zeta^2)^2} + O\left(\frac{a}{l}\right)^3, \tag{6.6}$$

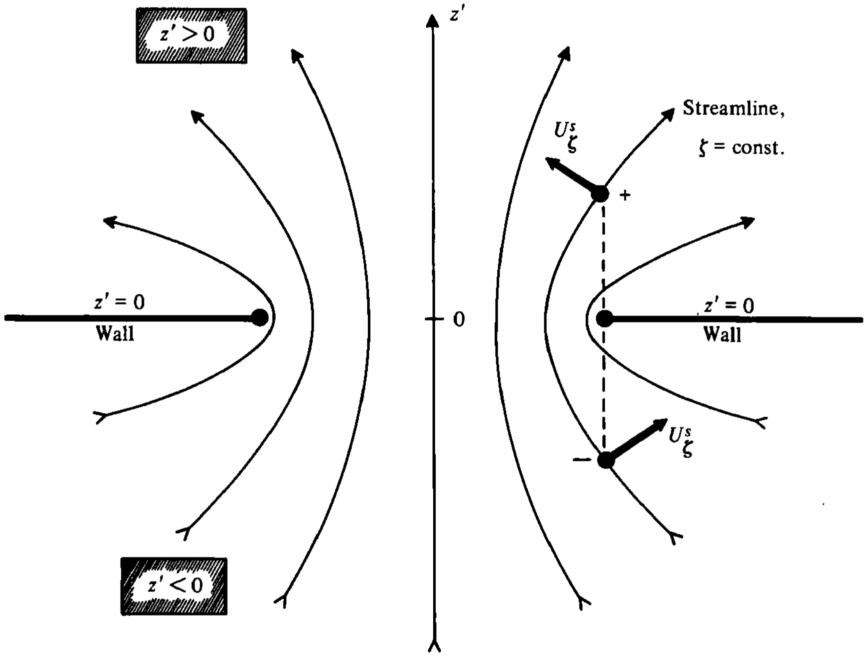

FIGURE 5. Migration of a spherical particle across the streamlines $\zeta = $ const. of the undisturbed flow. Net flow of fluid is in the positive z' direction. The direction of migration across the streamlines is shown for two mirror-image points, labelled + and −, corresponding, respectively, to the pair of values $(|\lambda|, \zeta)$ and $(-|\lambda|, \zeta)$, situated at equal distances above and below the plane $z' = 0$ of the hole.

in which (λ, ζ) are the co-ordinates of the sphere centre. By way of example, along the streamline $\zeta = 1$ (the z axis) (6.5) reduces to

$$U^s_\lambda = -\left(\frac{a}{c}\right)^2 \frac{\overline{V}}{(\lambda^2+1)^2}, \tag{6.7}$$

while along the streamline $\zeta = 0$ (i.e. along the solid portion of the wall),

$$U^s_\lambda = \left(\frac{a}{c}\right)^2 \frac{\overline{V}}{2|\lambda|^3(\lambda^2+1)^{\frac{3}{2}}}. \tag{6.8}$$

A negative value of U^s_λ implies that the sphere moves more slowly along the streamline than does the fluid, and conversely. That (6.7) and (6.8) possess different algebraic signs demonstrates that, contrary to intuition regarding the retarding effects of the wall upon the particle motion, the sphere does not always lag behind the fluid along a streamline. Rather, there exist regions in which the particle velocity actually exceeds that of the fluid. The equation $f(\lambda, \zeta) = 0$ of the surface separating these two regions corresponds to the vanishing of the square-bracketed term in (6.5), namely

$$\lambda^2 = (1-3\zeta^2)^{-1}\zeta^2(1+\zeta^2).$$

That U^s_ζ in (6.6) is non-zero indicates that the sphere migrates *across* the streamlines under the influence of the pressure gradient. The algebraic sign of U^s_ζ is positive for all $z' > 0$ and negative for all $z' < 0$, corresponding to the sketch in figure 5. Thus, upon approaching the hole the sphere is pushed away from the axis, while upon exiting from the hole the sphere is urged towards the axis.† Because of the symmetry of the flow about the plane $z' = 0$ the time-averaged migration of such a particle is necessarily zero with respect to equivalent positions on streamlines above and below the hole. Such symmetry would, however, be destroyed by particle–particle interactions, Brownian motion etc. Consequently, the implications of such migratory, motions upon the flow of a suspension remains an open question, worthy of further study.

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Appendix. Stokeslet near an opening in a wall backed by a bowl-shaped rigid boundary

Suppose now that, in contrast to the previous sections, the fluid region is

$$0 \leqslant \eta \leqslant \pi + \eta_1, \quad \xi \geqslant 0 \quad (0 < \eta_1 < \pi)$$

and consider the motion due to a stokeslet placed at $\rho = 0, z = z_0 = \cot \frac{1}{2}\eta_0 > 0$. This is the same problem as previously considered in §4, except that the fluid beyond the wall opening is now limited by a fixed bowl-shaped boundary. Because there cannot be any net flux across the orifice $\eta = \pi$, it is now helpful to work with a stream function ψ in terms of which the velocity components are given by (4.11). ψ must vanish at $\xi = 0$ and $\eta = 0, \pi + \eta_1$ and satisfy $L_{-1}^2 \psi = 0$ but care is still required in using the solution form (2.5) because of the difficulty, mentioned after (2.8), of satisfying the boundary conditions.

The solution ψ_0 for the case when the bowl opening is absent, leaving a solid plane ($\eta_1 = 0$), is given by

$$\psi_0 = \rho^2\{[(z-z_0)^2+\rho^2]^{-\frac{1}{2}} - [(z+z_0)^2+\rho^2]^{-\frac{1}{2}}\} \qquad \text{(A 1)}$$

$$+ 2\sqrt{2}\,\rho^2(\cosh\xi - \cos\eta)^{\frac{1}{2}} \sin\tfrac{1}{2}\eta_0 \sin\eta_0 \sin\eta \int_0^\infty \frac{\cosh s(\pi-\eta_0-\eta)}{\cosh s\pi} K_s'(\cosh\xi)\, ds$$

after using (2.6), (2.8) to ensure that

$$\psi_0 = 0 = \partial\psi_0/\partial\eta \quad \text{at} \quad \eta = 0, \pi. \qquad \text{(A 2)}$$

† That the direction of migration is different on the two sides of the hole may be seen directly from (5.9). Upon putting the arbitrary constant equal to zero we have that for a pair of mirror image points above and below the plane $z' = 0$,

$$p^+(\lambda, \zeta) = -p(-\lambda, \zeta).$$

Consequently,

$$\left.\frac{\partial p^+}{\partial \zeta}\right|_{\lambda,\zeta} = -\left.\frac{\partial p^+}{\partial \zeta}\right|_{-\lambda,\zeta},$$

or, equivalently,

$$\left.\frac{\partial p^+}{\partial \zeta}\right|_{\rho',z'} = -\left.\frac{\partial p^+}{\partial \zeta}\right|_{\rho',-z'}$$

Upon referring to figure 4, where the direction of increasing ζ is indicated by the unit vectors $\hat{\zeta}$, it is thus seen that, relative to the hole axis, the direction of the normal pressure gradients above and below the axis are such as to cause the sphere to migrate in opposite directions.

The solution ψ for $0 < \eta_1 < \pi$ can then be written

$$\psi = \psi_0 + 2\sqrt{2}\,\rho^2(\cosh\xi - \cos\eta)^{\frac{1}{2}} \sin\tfrac{1}{2}\eta_0 \int_0^\infty \frac{F_1(s,\eta)\,K_s'(\cosh\xi)}{(s^2+1)\cosh s\pi}\,ds \quad (0 \leqslant \eta \leqslant \pi),$$

and

$$\psi = 2\sqrt{2}\,\rho^2(\cosh\xi - \cos\eta)^{\frac{1}{2}} \sin\tfrac{1}{2}\eta_0 \int_0^\infty \frac{F_2(s,\eta)\,K_s'(\cosh\xi)}{(s^2+1)\cosh s\pi}\,ds \quad (\pi \leqslant \eta \leqslant \pi+\eta_1),$$

where

$$F_1(s,\eta) = A_1(s)\,[\sinh s\eta \cos\eta - s\cosh s\eta \sin\eta] + B_1(s)\sinh s\eta \sin\eta,$$
$$F_2(s,\eta) = A_2(s)\,[\sinh s(\pi+\eta_1-\eta)\cos(\pi+\eta_1-\eta) - s\cosh s(\pi+\eta_1-\eta)\sin(\pi+\eta_1-\eta)]$$
$$+ B_2(s)\sinh s(\pi+\eta_1-\eta)\sin(\pi+\eta_1-\eta).$$

This construction of F_1 and F_2 ensures that ψ and its η-derivative vanish at $\eta = 0, \pi+\eta_1$. It remains to determine the four functions $A_1(s)$, $A_2(s)$, $B_1(s)$, $B_2(s)$ by requiring that ψ and its first three η-derivatives be continuous at $\eta = \pi$. The equations obtained are

$$A_1\sinh s\pi + A_2(\sinh s\eta_1\cos\eta_1 - s\cosh s\eta_1\sin\eta_1) + B_2\sinh s\eta_1\sin\eta_1 = 0,$$
$$B_1\sinh s\pi + A_2(1+s^2)\sinh s\eta_1\sin\eta_1 - B_2(s\cosh s\eta_1\sin\eta_1 + \sinh s\eta_1\cos\eta_1) = 0,$$
$$A_1(1+s^2)\sinh s\pi - 2sB_1\cosh s\pi + A_2(1+s^2)(s\cosh s\eta_1\sin\eta_1 + \sinh s\eta_1\cos\eta_1)$$
$$- B_2[(s^2-1)\sinh s\eta_1\sin\eta_1 + 2s\cosh s\eta_1\cos\eta_1] = 2s(1+s^2)\sin\eta_0\sinh s\eta_0,$$
$$2sA_1(1+s^2)\cosh s\pi - (3s^2-1)B_1\sinh s\pi$$
$$- A_2(1+s^2)[(s^2-1)\sinh s\eta_1\sin\eta_1 + 2s\cosh s\eta_1\cos\eta_1]$$
$$+ B_2[(s^3-3s)\cosh s\eta_1\sin\eta_1 + (3s^2-1)\sinh s\eta_1\cos\eta_1]$$
$$= 2s(s^2+1)(s\cosh s\eta_0\sin\eta_0 - \sinh s\eta_0\cos\eta_0),$$

and their solution is

$$\Delta A_1 = s^3\sin\eta_0\sin^2\eta_1\cosh s(\pi-\eta_0) + \sinh s\eta_1\sinh s(\pi+\eta_1)\sinh s\eta_0\cos\eta_0$$
$$+ s^2\sin\eta_1(\sin\eta_0\cos\eta_1\cosh s\eta_0\sinh s\pi - \sin\eta_1\cos\eta_0\sinh s\eta_0\cosh s\pi)$$
$$- s\sinh s\pi\sinh s\eta_0\sin\eta_1\cos(\eta_0-\eta_1) - s\sinh s(\pi+\eta_1)\cosh s\eta_0\sinh s\eta_1\sin\eta_0,$$

$$\frac{\Delta B_1}{1+s^2} = s^2\sin^2\eta_1\sin\eta_0\sinh s(\pi-\eta_0) + s\sin\eta_1\sin(\eta_0-\eta_1)\sinh s\eta_0\sinh s\pi$$
$$+ \sin\eta_0\sinh s(\pi+\eta_1)\sinh s\eta_1\sinh s\eta_0,$$

$$\frac{\Delta A_2}{\sinh s\pi} = s^2\sin\eta_0\sin\eta_1\cosh s(\pi+\eta_1-\eta_0) - \sinh s(\pi+\eta_1)\sinh s\eta_0\cos(\eta_0-\eta_1)$$
$$- s\sin\eta_1\cos\eta_0\sinh s\eta_0\cosh s(\pi+\eta_1) + s\cos\eta_1\sin\eta_0\cosh s\eta_0\sinh s(\pi+\eta_1),$$

$$\frac{\Delta B_2}{(1+s^2)\sinh s\pi} = \sinh s\eta_0\sinh s(\pi+\eta_1)\sin(\eta_0-\eta_1) + s\sin\eta_0\sin\eta_1\sinh s(\pi+\eta_1-\eta_0),$$

where $\Delta = s^2\sin^2\eta_1 - \sinh^2 s(\pi+\eta_1)$.

The limit $\eta_1 \to \pi$ corresponds to the case where the radius of spherical bowl is infinite. In this situation, the flow is the superposition of two flows, one described by the solution given in §4, in which there is a net flow through the hole but zero pressure difference at infinity, and the other described by the solution given in Happel & Brenner in which there is both a net flow through the hole and a net pressure difference at infinity. The superposition of these two flows yields zero net flow through the hole with a non-zero pressure difference at infinity.

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